

# Spline Intersection Improvement

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## 1. Introduction

Rendering and simulation software needs many models of reality. Every human has hair and we need to visualize realistic hair. We can model hair with many spline curves. A typical task of the ray tracing method (see [2]) is finding an intersection of spline curves with a ray. We try to find a fast way to calculate the point where the ray intersects the curve.

### Model

- We have to model spline curve primitives with 4 control points in 3D space:  $\vec{p}_0; \vec{p}_1; \vec{p}_2; \vec{p}_3$ .
- Each control point has its own width of the curve:  $w_0; w_1; w_2; w_3$ .
- Spline curve center as function of curve evolution parameter is described by

$$\vec{p}(u) = \vec{p}_3 u^3 + 3\vec{p}_2 u^2(1-u) + 3\vec{p}_1 u(1-u)^2 + \vec{p}_0(1-u)^3.$$

- The width of primitive as function of the same evolution parameter is given by:

$$w(u) = w_3 u^3 + 3w_2 u^2(1-u) + 3w_1 u(1-u)^2 + w_0(1-u)^3.$$

The point that lies on the surface of the primitive satisfies the system

$$\begin{cases} |\vec{s}(u) - \vec{p}(u)|^2 = w(u)^2 \\ (\vec{s}(u) - \vec{p}(u)) \cdot \frac{d\vec{p}(u)}{du} = 0. \end{cases}$$

We point out that here  $w(u)$  means not the diameter but the radius of the primitive in the point  $\vec{p}(u)$ .

## 2. The Problem

Our task is the following:

For a given ray

$$\vec{r}(t) = \vec{o} + t\vec{d}$$

find  $t > 0$  and  $u \in [0, 1]$  such that

$$\vec{r}(t) = \vec{s}(u).$$

In case of multiple solutions, we are interested in the one that has minimum  $t$ .

### 2.1. Summary of the Report

- We first give an analytical solution of the problem. This leads us to Finding the roots of a function of two arguments.
- If we know that a certain ray intersects a single hair, we propose a simple iterative algorithm to find an approximation of the first point of intersection with any given accuracy.
- Having an iterative algorithm for some good cases we deal with some exceptional ones.
- In the end we consider the problem of whether there is an intersection between a given ray and a certain hair.

#### Notations

The surface of the hair is given by  $\vec{s}(u, v)$  where  $u \in [0, 1]$ ,  $v \in [0, 2\pi)$ :

$$(2.1) \quad \vec{s}(u, v) = \vec{p}(u) + w(u) \cdot \cos(v) \cdot \vec{a} + w(u) \cdot \sin(v) \cdot \vec{b}.$$

Components of the vectors are:

$$\begin{aligned} \vec{d} &= (d_x, d_y, d_z)^\top, \\ \vec{o} &= (o_x, o_y, o_z)^\top, \\ \vec{p} = \vec{p}(u) &= (p_x, p_y, p_z)^\top, \\ \frac{d\vec{p}(u)}{du} \quad || \quad \vec{q} = \vec{q}(u) &= (q_x, q_y, q_z)^\top, \\ \vec{a} = \vec{a}(u) &= (a_x, a_y, a_z)^\top, \\ \vec{b} = \vec{b}(u) &= (b_x, b_y, b_z)^\top, \\ \vec{r} = \vec{r}(t) &= (r_x, r_y, r_z)^\top, \\ \vec{s} = \vec{s}(u, v) &= (s_x, s_y, s_z)^\top. \end{aligned}$$

**Remark 1.** The vectors  $\vec{d}$ ,  $\vec{q}$ ,  $\vec{a}$ ,  $\vec{b}$  are normed, i.e.  $\|\cdot\| = 1$ . More,  $\vec{q}$ ,  $\vec{a}$ ,  $\vec{b}$  are orthogonal and  $\vec{b} = \vec{q} \times \vec{a}$ . The following rule applies to the selection of  $\vec{a}$ :

If  $|q_x| \leq \min\{|q_y|, |q_z|\}$  then  $\vec{a} = (0, q_z, -q_y)^\top$  and  $\vec{b} = (-q_x^2 - q_z^2, q_x q_y, q_x q_z)^\top$ ;

If  $|q_y| \leq \min\{|q_x|, |q_z|\}$  then  $\vec{a} = (q_z, 0, -q_x)^\top$  and  $\vec{b} = (-q_x q_y, q_x^2 + q_z^2, -q_y q_z)^\top$ ;

If  $|q_z| \leq \min\{|q_y|, |q_x|\}$  then  $\vec{a} = (q_y, -q_x, 0)^\top$  and  $\vec{b} = (q_x q_z, q_y q_z, -q_x^2 - q_y^2)^\top$ ;

**Remark 2.** For the ray  $\vec{r}(t)$  and a fixed  $u$  we have two cases.

Case 1: If the ray  $\vec{r}(t)$  is not parallel to the plane defined by the point  $\vec{p}(u)$  and the vector  $\vec{q}(u)$  which is orthogonal to this plane (we note that  $\vec{q}(u)$  is the tangent of the curve described by the center of the hair  $\vec{p}(u)$ ), then we can calculate

$$(2.2) \quad t = \frac{(\vec{q}(u), \vec{p}(u) - \vec{o})}{(\vec{q}(u), \vec{d})} \quad \text{and,} \quad \vec{r}(t) = \vec{o} + t\vec{d}, \quad \delta = \text{dist}(\vec{r}(t), \vec{p}(u)) - w(u).$$

Case 2:  $\vec{d} \perp \vec{q}(u)$  and  $(\vec{q}(u), \vec{d}) = 0$ , we calculate only  $(\vec{q}(u), \vec{p}(u) - \vec{o})$  which gives us the signed distance between the ray and the plane - we need this scalar product to be 0.

### Analytical solution of the problem

An intersection between surface  $\vec{s}(u, v)$  and the ray  $\vec{r}(t)$  happens in a point  $(u, t)$  which is a solution of the system:

$$\vec{s}(u, v) = \vec{r}(t)$$

or written in components ( $u$  is omitted for simplicity):

$$(2.3) \quad \begin{cases} p_x + w \cdot \cos(v) \cdot a_x + w \cdot \sin(v) \cdot b_x & = & o_x + t \cdot d_x \\ p_y + w \cdot \cos(v) \cdot a_y + w \cdot \sin(v) \cdot b_y & = & o_y + t \cdot d_y \\ p_z + w \cdot \cos(v) \cdot a_z + w \cdot \sin(v) \cdot b_z & = & o_z + t \cdot d_z \end{cases}$$

or equivalently

$$\begin{cases} w \cdot \cos(v) \cdot a_x + w \cdot \sin(v) \cdot b_x & = & o_x + t \cdot d_x - p_x \\ w \cdot \cos(v) \cdot a_y + w \cdot \sin(v) \cdot b_y & = & o_y + t \cdot d_y - p_y \\ w \cdot \cos(v) \cdot a_z + w \cdot \sin(v) \cdot b_z & = & o_z + t \cdot d_z - p_z. \end{cases}$$

After we square and sum up the three equations above we get:

$$\begin{aligned} w^2 \cdot \cos^2(v) \cdot (a_x^2 + a_y^2 + a_z^2) + w^2 \cdot \sin^2(v) \cdot (b_x^2 + b_y^2 + b_z^2) \\ + 2 \cdot w^2 \cdot \cos(v) \cdot \sin(v) \cdot (a_x b_x + a_y b_y + a_z b_z) = \end{aligned}$$

$$\begin{aligned}
&= t^2 \cdot (d_x^2 + d_y^2 + d_z^2) - 2t \cdot ((p_x - o_x) \cdot d_x + (p_y - o_y) \cdot d_y + (p_z - o_z) \cdot d_z) \\
&\quad + (p_x - o_x)^2 + (p_y - o_y)^2 + (p_z - o_z)^2, \\
&\quad t^2 - 2t \cdot \vec{d} \cdot (\vec{p} - \vec{o}) + (\vec{p} - \vec{o})^2 - w^2 = 0.
\end{aligned}$$

We now substitute:  $B = \vec{d} \cdot (\vec{p} - \vec{o}) = (\vec{d}, \vec{p} - \vec{o})$ ,  $C = (\vec{p} - \vec{o}, \vec{p} - \vec{o}) - w^2$  and receive the quadratic equation:

$$(2.4) \quad t^2 - 2.B.t + C = 0.$$

The solution depends from

$$(2.5) \quad D = (B^2 - C).$$

If  $D > 0$  we have two roots:  $t_1 = B + \sqrt{D}$ ,  $t_2 = B - \sqrt{D}$ .

If  $D = 0$  we have one root:  $t_0 = B$ .

If  $D < 0$  we have no roots.

Let  $t_0$  be a solution of (2.4) and we know  $\vec{r}(t_0)$ ,  $\vec{r}(t_0) - \vec{p}(u)$ . We have to check whether they are foreign solutions:

Let  $\varepsilon > 0$  be some tolerance. If  $|(\vec{q}, \vec{r} - \vec{p})| > \varepsilon$  then this  $t_0$  is not a solution of the system (2.3).

Now equation (2.4) is a of the type  $F(u, t) = 0$ . The highest degree of  $u$  is 6 and of  $t$  is 2. Since we know that  $u \in [0, 1]$  and also know the cuboid which contains the hair primitive (the company that has proposed the problem has information about it), we are able to restrict  $t$  within a certain interval  $[T_0, T_N]$ . This reduces the problem to finding the zeros of the function  $F(u, t)$  in the rectangle  $[0, 1] \times [T_0, T_N]$ , for which we can apply some known algorithm. From the obtained (if any) zeros in that rectangle (which are also a solution of the system (2.3)) we chose the one with the smallest  $t$ .

## 2.2. An iterative method

If we know that there is an intersection between the hair and the ray that we are considering, we divide the interval  $[0, 1]$  into  $N$  subintervals with the points  $u_i = i/N$ ,  $i = 0, \dots, N$ . For a fixed  $u_i$  we consider the plane that is orthogonal to the vector of the direction  $\frac{d\vec{p}(u_i)}{du}$  and passes through the point  $\vec{p}(u_0)$ . We are interested in the point in which the ray  $\vec{r}(t)$  intersects this plane. Let this happen for  $t = t_i$ . Then the vector  $\vec{r}(t_i) - \vec{p}(u_i)$  must be orthogonal to the direction of the curve describing the center of the hair, i.e. we have

$$(\vec{r}(t_i) - \vec{p}(u_i)) \cdot \frac{d\vec{p}(u_i)}{du} = (\vec{o} + t_i \vec{d} - \vec{p}(u_i)) \cdot \frac{d\vec{p}(u_i)}{du} = 0,$$

or equivalently

$$(o_x + t_i d_x - p_x(u)) \frac{dp_x(u)}{du} + (o_y + t_i d_y - p_y(u)) \frac{dp_y(u)}{du} + (o_z + t_i d_z - p_z(u)) \frac{dp_z(u)}{du} = 0,$$

which is a linear equation for  $t_i$ . Having the point from the ray  $\vec{r}(t_i)$ , we calculate how far it is from the point  $\vec{p}(u_i)$ :

$$|\vec{r}(t_i) - \vec{p}(u_i)|^2 = [t_i d_x - p_x(u_i)]^2 + [t_i d_y - p_y(u_i)]^2 + [t_i d_z - p_z(u_i)]^2.$$

When we have that

$$|\vec{r}(t_{i-1}) - \vec{p}(u_{i-1})|^2 > w(u_{i-1})^2$$

and

$$|\vec{r}(t_i) - \vec{p}(u_i)|^2 < w(u_i)^2$$

for some  $u_{i-1}$  and  $u_i$ , this means that somewhere between  $u_{i-1}$  and  $u_i$  (correspondingly  $t_{i-1}$  and  $t_i$ ) the ray intersects the hair. Actually we have to find all such couples of points  $(u_{i-1}, u_i)$  and then take the smallest corresponding  $t_{i-1}$  and  $t_i$  (this is because we may have several intersections and it is possible to have the smallest  $t$  for the largest  $u$ ). Then we can break the interval  $[u_{i-1}, u_i]$  into subintervals with the points  $v_0, \dots, v_{N_1}$  and repeat the algorithm for the interval  $[v_0, v_{N_1}]$  instead of  $[0, 1]$ . We repeat this algorithm until the distance between  $\vec{r}(t_{i-1})$  and  $\vec{r}(t_i)$  becomes less than the required accuracy. This approach is being illustrated on Fig. 1 and Fig. 2.

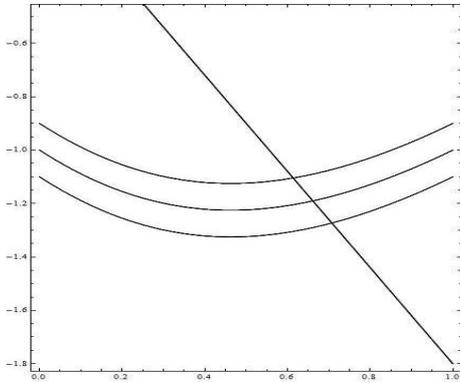


Fig. 1a. The central curve is the center of the hair and the outer are from its surface

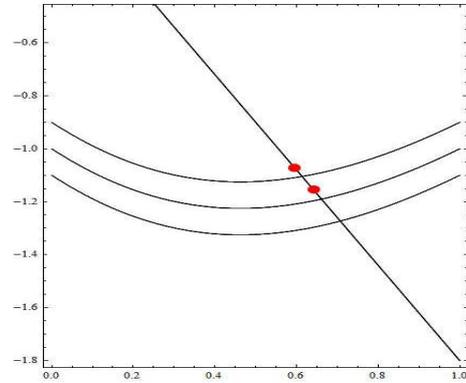


Fig. 1b. We have two points  $\vec{r}(t_{i-1})$  and  $\vec{r}(t_i)$  from the ray that correspond to two consecutive points from the curve of the center of the hair.

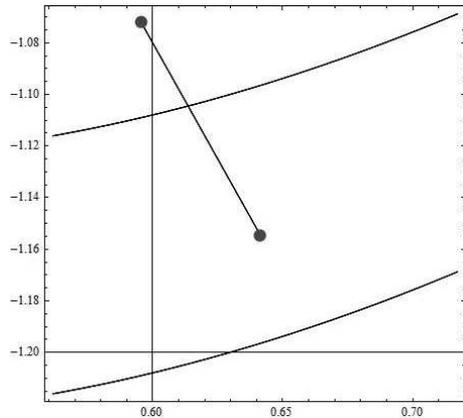


Fig. 2a. Now we consider the part of the ray which is between the points  $\vec{r}(t_{i-1})$  and  $\vec{r}(t_i)$ .

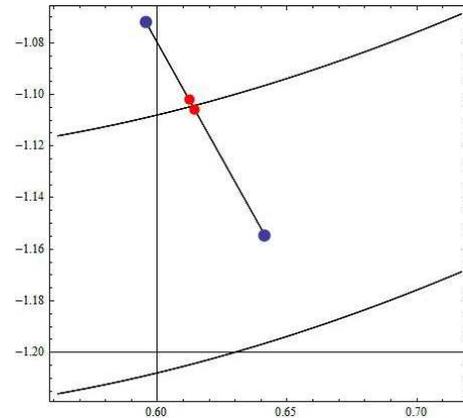


Fig. 2b. We get the two inner points when we apply again our method for the subinterval  $[u_{i-1}, u_i]$  which is divided in 10 subintervals.

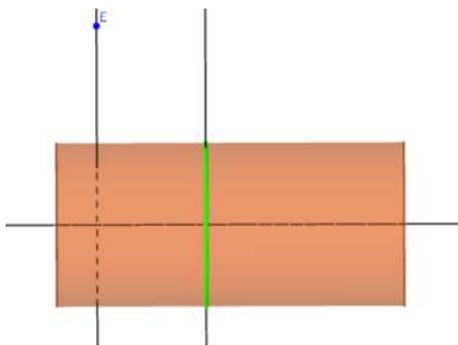


Fig. 3a. The ray is parallel to the plane that passes through  $\vec{p}(u)$  and is orthogonal to  $\vec{q}(u)$ .

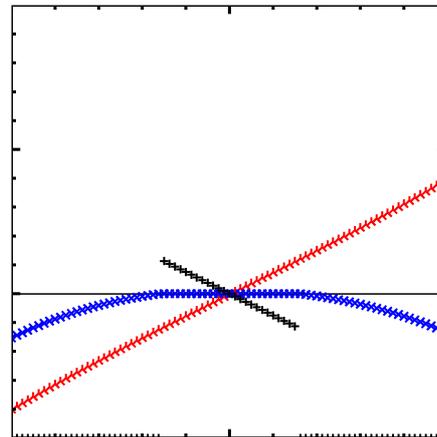


Fig. 3b

### 2.3. Exceptions

#### A plane parallel to the ray

A problem with the algorithm described above will occur if the fixed plane through  $\vec{p}(u)$  and orthogonal to  $\frac{d\vec{p}}{du}(u)$  is parallel to  $\vec{d}$  (see Figure 3a). This means that we have  $(\vec{q}(u), \vec{d}) = 0$ . This case was tackled in case 2 of Remark 2.

### A short description of Fig. 3b, Fig. 4b and Fig. 5b

The horizontal axis corresponds to the variable  $u \in [0, 1]$ . In our numerical experiments the step is 0.005 (see also Section 2.2 for this approach). The vertical axis has several meanings:

- The curve with symbols “X” (blue in color variant) responds to the value  $D$  from (2.5) if  $D < 0$ , and  $\text{dist}(\vec{p}(u), \vec{r}(t)) - w(u)$  otherwise, where  $t$  is solution of (2.4).

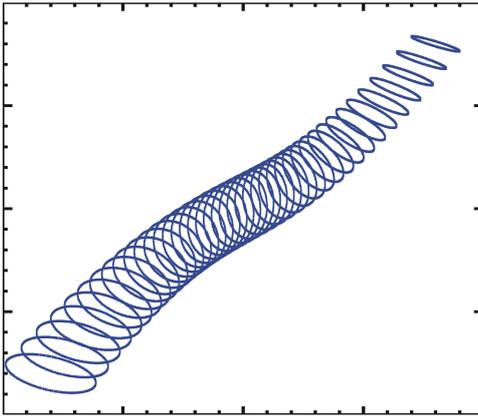


Fig. 4a

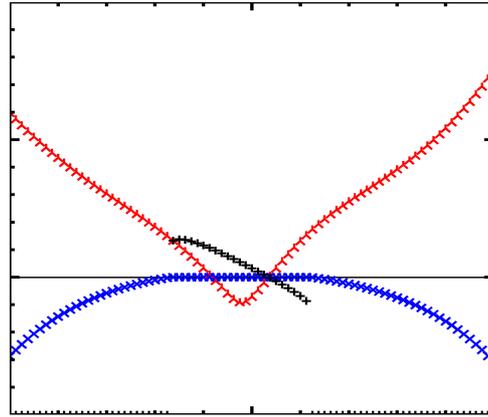


Fig. 4b

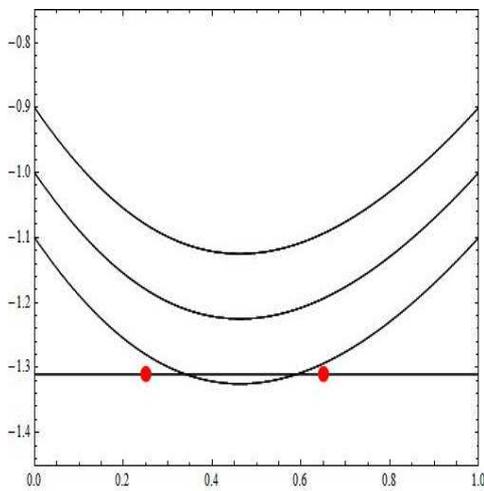


Fig. 5a

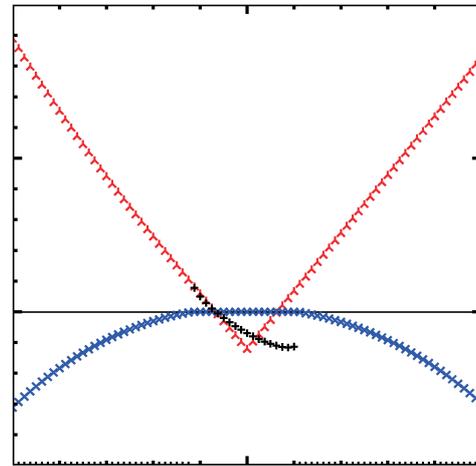


Fig. 5b

- The angled linetype with rotated symbol “Y” (red in color variant) corresponds to the  $\delta$  from (2.2) in Case 1 of Remark 2 if  $(\vec{q}(u), \vec{d}) \neq 0$  and to the value  $(\vec{q}(u), \vec{p}(u) - \vec{o})$  from Case 2 of the Remark 2 otherwise.
- The linetype with symbols “+” (black in color variant) is result of checking whether the solution of (2.4) is solution for (2.3), i.e. checking orthogonality of  $\vec{q}$  and  $(\vec{r} - \vec{p})$ .

The range of vertical axis is  $[-0.5, 1.0]$  and the horizontal straight line is the zero (i.e. passes through zero).

For the more general case (see Fig. 4a for hair and Fig. 4b for the analysis) for all  $u$  the “Y” curve describes the first case of Remark 2. Simultaneously, the localization by proposed method above for the problem (2.3) also works well (see “X” line and short “+” one in the middle on Fig. 4b).

For the special case when the central curve of the hair lies in a plane perpendicular to the ray, then orthogonal planes to the central curve for all  $u$  are parallel to the ray (see Fig. 3a), but that does not stop us to locate the intersection of the beam  $\vec{r}(t)$  with the surface  $\vec{s}(u, v)$ . Note that all points from “Y”-curve for Fig. 3b are from second case of Remark 2.

Fig. 5b illustrates another particular case, when the beam pierces the surface near the border at the intersection may be omitted.

*How close we are to the intersection?* The calculations from Remark 2 give an answer to this question.

### When the step is too big

A possible problem when applying the iterative algorithm is when a very little segment of the line described by  $\vec{r}(t)$  is in the hair. In such cases it is possible to have two points of the ray  $\vec{r}(t_i)$  and  $\vec{r}(t_{i+1})$  both of which are outside the hair but between them there is an intersection (see Figure 5a). If we know that there should be an intersection, but we don't find one, we may divide the interval of the parameter  $u$  into more subintervals.

Another approach to deal with such cases is to look for “suspicious” points, i.e. points  $u_i$  and  $t_i$  for which

$$R_i := |\vec{r}(t_i) - \vec{p}(u_i)|^2 - w(u_i)^2$$

is less than a certain (small) value. We will estimate  $R_i + R_{i+1}$ . The mentioned estimate of  $R_i + R_{i+1}$  we calculate when the ray is tangential to the primitive of the hair at one end point for  $[u_i, u_{i+1}]$ , say  $u_i$ . In addition, we assume the maximal perturbation of the axis  $p(u)$  in the opposite direction of  $B_{i+1}$  (the geometrical

meaning of  $B_i = B(u_i)$  is the common point of the ray and the plane through  $\vec{p}(u_i)$  and perpendicular to  $\vec{q}(u_i)$  and we approximate it by  $\frac{1}{2}|p''|h^2$ . Similarly, we assume the maximal change of  $w(u)$  from its tangent and also approximate it up to second order. Then, for the criterion of this exception we obtain

$$R_i + R_{i+1} < \epsilon, \quad (R_i, R_{i+1} > 0),$$

where

$$\epsilon = h^2[(|\vec{q}| \tan |\varphi| + \frac{1}{2}|\vec{k}|h)^2 + (|\vec{k}| + |w''|)w]$$

with  $\vec{q} = \frac{d\vec{p}}{du}(u_i)$  (or  $u_{i+1}$ ),  $\vec{k} = \frac{d^2\vec{p}}{du^2}(u_i)$ ,  $\varphi = \angle(\vec{d}, \vec{q})$ ,  $w = w_i$  (or  $w_{i+1}$ ) and  $w'' = w''(u_i)$ .

We see that the probability for this exception is "very small" for "small"  $h$ .

#### 2.4. Investigation if the ray intersects the hair

Now we consider the question of whether the ray intersects or not the hair primitive. An important aspect of our approach here is that the company has the information about the cuboid which contains the primitive (Fig. 6). This

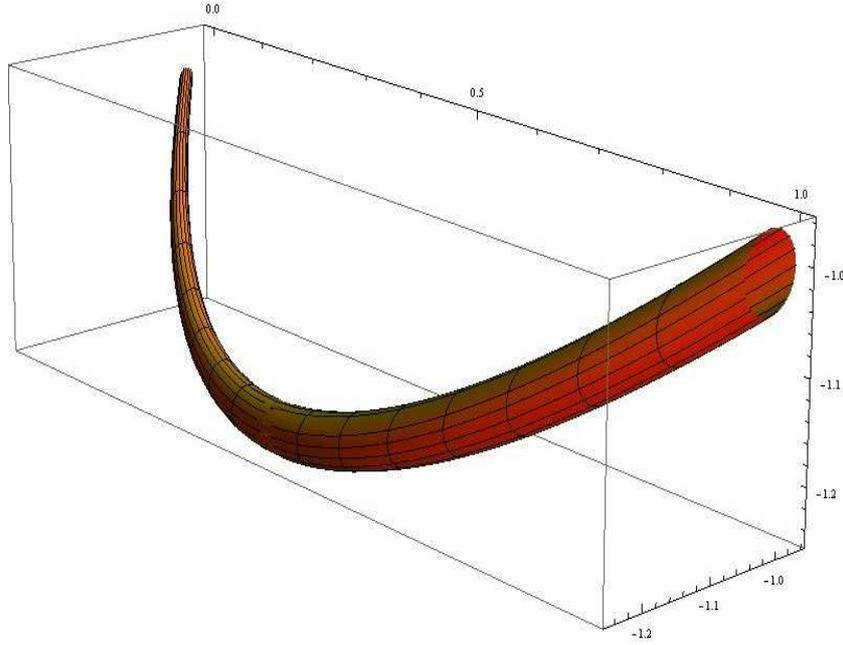


Fig. 6. The cuboid which contains the considered hair primitive is known.

means that we are able to define the interval  $[T_0, T_N]$  for which  $\vec{r}(t)$  belongs to the cuboid. Without loss of generality we may consider  $\vec{d}$  as a unit vector. If the ray intersects the hair then there is a point  $(u_0, t_0)$  (many points but one is enough) for which

$$|\vec{o} + t_0\vec{d} - \vec{p}(u_0)|^2 < w(u_0)^2$$

or equivalently

$$|\vec{o} + t_0\vec{d} - \vec{p}(u_0)|^2 - w(u_0)^2 < 0.$$

So, we are interested whether the expression

$$t^2 - 2Bt + C$$

where  $B = B(u) = \vec{d} \cdot (\vec{p} - \vec{o}) = (\vec{d}, \vec{p} - \vec{o})$  and  $C = c(u) = (\vec{p} - \vec{o}) - w(u)^2$  becomes less than 0 in the rectangle  $[0, 1] \times [T_0, T_N]$ . This reduces the problem to finding the minimum of a function of two variables  $t$  and  $u$  – the highest degree of  $u$  is 6 and of  $t$  is 2. Actually this is the expression obtained when we derived the analytical solution with the difference that here we are interested not in its zeros, but in its absolute minimum.

### 3. Summary

The problem was to find the first point of intersection (or an approximation of this point) between a ray and a hair in the fastest possible way. We did the following activities:

- found an analytical solution of the problem;
- developed an iterative algorithm for finding the point of intersection (when we know we have one);
- considered some exceptional cases;
- made up an algorithm for finding whether a given ray intersects the hair we are considering.

One can choose from two options: numerically find the zeros (if any) of a function of two variables (one of degree 2 and the other of degree 6) or numerically find its minimum and than (if this minimum is less than zero) use the iterative approach.

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